HYDRO- AND THERMODYNAMICS OF TORNADOES AND OCEANIC WATERSPOUTS

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There exist several approaches to description of the tornado. The air column from the earth is modelled by a turbulent filament, acting with a given intensity perpendicular to a rigid wall [1]. The air column from a storm cloud is described by use of certain additional magnetohydrodynamic hypotheses [2]. In recent times a number of numerical models have been constructed on the basis of exact Navier–Stokes equations [3-5]. In the present study a modified formulation of the Cauchy–Poisson problem of motion of a heavy viscous incompressible fluid with a surface having a discontinuity in density will be used with two simplifying assumptions: 1) the major inertial force in the column is characterized by a transverse acceleration; 2) upon the density discontinuity surface the vertical gradient of the transverse velocity and the temperature perturbations characterizing the temperature change between this surface and the deep fluid layers are specified. Normal vertical and tangent radial stresses are absent on this surface. The density discontinuity surface may be a water surface, the boundary of an atmospheric cloud, or a boundary marked by a sharp change in air density, produced by a storm, volcanic eruption, etc. Evidence in favor of these assumptions is found in numerous radar observations of tornadoes, which demonstrate directly that the mechanical and convective motions which form the air column are of a turbulent nature [6, 7]. These turbulences and certain other factors (oppositely directed wind flows at the density discontinuity surface, electromagnetic effects, thermodynamic causes, etc.) indicate that the main inertial force in the air column is transverse inertia, with the air column itself developing as a result of interaction of the density discontinuity surface with turbulent air flows above this surface. This interaction can be modelled by a vertical transverse velocity gradient on the discontinuity surface. The cause of the vertical gradient remains unspecified, and it is possible to consider indirectly the entire sum of factors enumerated above, which apparently [8] cause formation of the tornado. We note that total rejection of inertial forces greatly distorts the tornado model, permitting (for a specified turbulent load) determination of only one transverse velocity in this complex spatial flow [9]. For simplicity we neglect the effect of the earth's surface upon the tornado. Then atmospheric tornadoes and oceanic waterspouts will differ only in the values of the hydro- and thermodynamic parameters, and can both be described by the following method.

We assume that a heavy viscous incompressible fluid occupies the volume of space $z' < \zeta'(r)$ in a cylindrical coordinate system r', θ, z' with origin on the unperturbed surface $z'=0$ and axis Oz, directed opposite the force of gravity. The fluid moves under the influence of a vertical transverse velocity gradient $\mu^{-1}\omega'(r')$ on the density discontinuity surface $z' = \zeta'$ and a temperature differential $\vartheta'(r')$ between this surface and the deep layers of the fluid. The quantity ω' in this formulation has the dimensions of stress. It is physically equivalent to the transverse tangent stress on a horizontal area intersecting the density discontinuity surface. We will assume that other stresses on this area are absent. The problem of establishing the steady-state motion of the fluid under the action of these factors requires determination of the velocity $\mathbf{v} = \{v_r, v_\theta, v_z\}$, the difference $p' - p_*$ between hydrodynamic and atmospheric pressures, the form of the surface ζ' and the difference T' between the temperature at a given point and the temperature as $z' \to -\infty$ as functions of r' and z' from the system of Navier–Stokes and thermal balance equations (with consideration of the Archimedean buoyant force)

$$(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\left(\frac{p' - p_*}{\rho} + gz'\right) = -\beta T'\mathbf{g} + \nu\nabla^2\mathbf{v},\ \nabla\mathbf{v} = 0, \tag{1}$$

$$\rho c\left(v_r\frac{\partial T'}{\partial r'} + v_z\frac{\partial T'}{\partial z'}\right) = \lambda\left[\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial T'}{\partial r'}\right) + \frac{\partial^2 T'}{\partial z'^2}\right] + \mu\left\{2\left[\left(\frac{\partial v_r}{\partial r'}\right)^2 + \left(\frac{v_r}{r'}\right)^2 + \left(\frac{\partial v_z}{\partial z'}\right)^2\right] + \left[r'\frac{\partial}{\partial r'}\left(\frac{v_\theta}{r'}\right)\right]^2 + \left(\frac{\partial v_\theta}{\partial z'}\right)^2 + \left(\frac{\partial v_r}{\partial z'} + \frac{\partial v_z}{\partial r'}\right)^2\right\},$$

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with boundary conditions on the surface $z' = \zeta'$

$$-(p' - p_*) + 2\mu \frac{\partial v_z}{\partial z'} = 0, \quad \frac{\partial v_z}{\partial r'} + \frac{\partial v_r}{\partial z'} = 0, \quad T' = \vartheta'(r'), \tag{2}$$
$$\mu \frac{\partial v_\theta}{\partial z'} = -\omega'(r'), \quad v_z = v_r \frac{\partial \zeta'}{\partial r'}$$

and with the condition that v_r, v_θ, and T' disappear as $z' \to -\infty$. The density ρ, the kinematic viscosity ν, $\mu = \rho\nu$, specific heat c, thermal conductivity λ, and coefficient of volume thermal expansion β of the fluid are considered constant. We take

$$\Omega = \max_{r'} |\omega'(r')|, \quad \omega = \frac{\omega'}{\Omega}, \quad p' = p_* + \frac{\rho g^2 \mu^2}{\Omega^2}(p - z), \quad T = \beta T', \tag{3}$$
$$v_r = \frac{\rho\mu^3 g^3}{\Omega^4} u, \quad v_\theta = \frac{g\mu}{\Omega} v, \quad v_z = \frac{\rho\mu^3 g^3}{\Omega^4} w, \quad \zeta' = \frac{\rho^2 g^5 \mu^6}{\Omega^8} \zeta,$$
$$r = \frac{r'\Omega^2}{g\mu^2}, \quad z = \frac{z'\Omega^2}{g\mu^2}, \quad R = \frac{\rho^2\mu^4 g^4}{\Omega^6}, \quad \gamma = \frac{\beta g^2 \mu^3}{\lambda\Omega^2}, \quad \vartheta = \beta\vartheta'.$$

With such a choice of dimensionless variables, upon substitution in Eqs. (1), (2) the nonlinear term v^2/r in the motion equation will have a coefficient of unity, while the remaining nonlinear inertial terms will have the small parameter R as their coefficient. By expanding the boundary conditions in power series in ζ, we obtain a system of equations and boundary conditions with a small parameter standing before the quadratic and higher nonlinear terms (except for v^2/r). We denote this system in the form $A(\bar{u}, R) = RB(\bar{u}, R)$, $\bar{u} = \{u, v, w, p, \zeta\}$. The operator A contains the term v^2/r only in its radial component. The expansion of the operator B in powers of the elements of the matrix $\bar{u}$ and their derivatives begins with quadratic terms. Therefore, in the operator equation thus obtained the method of successive approximations may be used, if unity is not an eigennumber of the operator L^{-1} (where L is the linear operator obtained from A by discarding the nonlinear term). We will assume that the parameter R is chosen such that unity is not within the spectrum of the operator L. Then for sufficiently small R, the successive approximation process will converge, with the first approximation comprising O(R) with respect to the zeroth approximation. For the zeroth approximation, to which we will limit ourselves in view of the smallness of R (for $\Omega \sim 1$ N/m², $R \sim 0.01$ for water), we obtain the following equations and boundary conditions: $A(\bar{u}, R) = 0$, i.e.,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0, \quad \left.\frac{\partial v}{\partial z}\right|_{z=0} = -\omega(r), \quad v|_{z=-\infty} = 0; \tag{4}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{v^2}{r} = \frac{\partial p}{\partial r}, \quad \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0; \tag{5}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} = \frac{\partial p}{\partial z} - T; \tag{6}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = -\gamma\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}; \tag{7}$$

$$R\zeta = p - 2\frac{\partial w}{\partial z}, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0, \quad w = 0, \quad T = \vartheta(r) \quad \text{for} \quad z = 0,$$
$$u = T = 0 \quad \text{for} \quad z = -\infty. \tag{8}$$

The dimensionless transverse velocity v defined by Eq. (4) coincides with that found in [10] in a study of flow without convection. It is equal to ($Q_{1/2}$ being a Legendre function)

$$v = -\frac{1}{\pi\sqrt{r}}\int_0^\infty \sqrt{x}\,\omega(x)\,Q_{1/2}\left(\frac{x^2 + r^2 + z^2}{2rx}\right)dx. \tag{9}$$

Integral representations of the solution of Eqs. (5)-(8) may be constructed with the aid of a Hankel transform and have the form

$$u = \int_0^\infty s u_1(s, z) J_1(rs)\, ds, \quad w = \int_0^\infty s w_0(s, z) J_0(rs)\, ds, \tag{10}$$

$$p = \int_0^\infty s p_0 (s, z) J_0 (rs)\, ds, \quad T = \int_0^\infty s T_0 (s, z) J_0 (rs)\, ds,$$

$$T_0 = \vartheta_0 e^{sz} + \frac{1}{2s} \int_0^\infty \varphi_0 (s, -x) \left[e^{s(z-x)} - e^{-s|z+x|}\right] dx,$$

$$\vartheta_0 = \int_0^\infty r \vartheta (r) J_0 (rs)\, dr, \quad \varphi_0 = -\gamma \int_0^\infty r \left\{\left[r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\} J_0 (rs)\, dr,$$

$$p_0 = -\frac{1}{2} \int_0^\infty T_0 (s, -x) \left[e^{s(z-x)} - \operatorname{sgn}(z + x)\, e^{-s|z+x|}\right] dx -$$

$$-\frac{1}{2} \int_0^\infty f_1 (s, -x) \left[e^{s(z-x)} + e^{-s|z+x|}\right] dx, \quad f_1 = \int_0^\infty v^2 (r, z) J_1 (rs)\, dr,$$

$$u_1 = \frac{1}{4s} \int_0^\infty T_0 (s, -x) \left\{[2 + s (z - x)]\, e^{s(z-x)} + s (z + x)\, e^{-s|z+x|}\right\} dx +$$

$$+ \frac{1}{4s} \int_0^\infty f_1 (s, -x) \left\{[1 + s (z - x)]\, e^{s(z-x)} + [1 - s\, |z + x|]\, e^{-s|z+x|}\right\} dx,$$

$$w_0 = -\frac{1}{4s} \int_0^\infty T_0 (s, -x) \left\{[1 - s (z - x)]\, e^{s(z-x)} - [1 + s\, |z + x|]\, e^{-s|z+x|}\right\} dx -$$

$$-\frac{1}{4} \int_0^\infty f_1 (s, -x) \left[(z - x)\, e^{s(z-x)} + (z + x)\, e^{-s|z+x|}\right] dx,$$

$$\zeta = \zeta^{(1,1)} + \zeta^{(1,2)} + \zeta^{(2)}, \quad \zeta^{(1,1)} = \frac{1}{2R} \int_0^\infty \vartheta_0 (s) J_0 (rs)\, ds,$$

$$\zeta^{(1,2)} = \frac{1}{R} \int_0^\infty \int_0^\infty \frac{1}{s} \left(1 - e^{-xs}\right) \varphi_0 (s, -x) J_0 (rs)\, dx ds,$$

$$\zeta^{(2)} = \frac{1}{R} \int_0^\infty \int_0^\infty x s^2 e^{-xs} f_1 (s, -x) J_0 (rs)\, dx ds.$$

Just like the form of the fluid density discontinuity surface, all the characteristics of the flow under consideration are composed of perturbations due to turbulent tangent stresses on the surface, the temperature differential, and perturbations connected with dissipation of mechanical energy.

The general formulas of Eq. (10) contain as a special case flows corresponding to atmospheric tornadoes and oceanic water spouts. To distinguish such flows, we assume that the turbulent tangent stresses $\omega'(r')$ and the temperature distribution $\vartheta'(r')$ on the surface $z' = \zeta'$ are nonzero over the ranges $r' \leqslant a'$ and $r' \leqslant b'$, where they have constant values $(-\Omega)$ and Θ'. In the variables of Eq. (3)

$$\omega(r) = -1 \text{ for } r \leqslant a, \ \omega(r) = 0 \text{ for } r > a, \ a = a'\Omega^2/g\mu^2; \tag{11}$$

$$\vartheta(r) = \beta\Theta' = \Theta \text{ for } r \leqslant b, \ \vartheta(r) = 0 \text{ for } r > b, \ b = b'\Omega^2/g\mu^2. \tag{12}$$

We will study the flow components produced by turbulent loading on the surface asymtotically, assuming that the dimensionless radius a of its field of action is sufficiently small. Then from Eqs. (9), (11) we find the transverse velocity, which as $a \to 0$ is equal to [10]

$$v = a^3 r / 6(r^2 + z^2 + a^2)^{3/2}. \tag{13}$$

Introducing this expression into Eq. (10), we find the Hankel transform for the transverse acceleration (f_1) and the dissipative function (φ_0) [11]

$$f_1 = \frac{a^6 s^2 K_1\left(s\sqrt{z^2+a^2}\right)}{288\sqrt{z^2+a^2}}; \quad (14)$$

$$\varphi_0 = -\frac{\gamma a^6 s^2}{192}\left[\frac{K_2\left(s\sqrt{z^2+a^2}\right)}{z^2+a^2} - \frac{a^2 s K_3\left(s\sqrt{z^2+a^2}\right)}{8\left(z^2+a^2\right)^{3/2}}\right]\left[1+O\left(\sqrt{z^2+a^2}\right)\right] \quad (15)$$

(where K_n is a Macdonald function of the second sort).

We introduce Eqs. (14), (15) into Eq. (10). We then obtain integrals which diverge at $a = 0$. Therefore the major contribution to their asymptote as $a \to 0$ is produced by the vicinity of the point $x = s = 0$, and we may set $e^{-xs} \sim 1$, $1 - e^{-xs} \sim xs$. On this path the smooth terms of the asymptote of the integrals of Eq. (10), determined by turbulent loading on the surface, are expressed in terms of elementary functions, while those related to energy dissipation are described by a hypergeometric series. Thus, for the corresponding terms of the surface ζ we obtain

$$\zeta^{(2)} = \frac{a^6\left(a^2-r^2\right)}{72R\left(a^2+r^2\right)^3}, \quad (16)$$

$$\zeta^{(1,2)} = -\frac{\pi\gamma a^3}{3072R}\left[6F\left(\frac{3}{2}, \frac{1}{2}; 1; -\frac{r^2}{a^2}\right) - F\left(\frac{3}{2}, \frac{3}{2}; 1; -\frac{r^2}{a^2}\right)\right].$$

From this and Eq. (3) it follows that the fraction of dissipative as compared to turbulent perturbations comprises $\frac{45\pi}{384}\frac{g\mu\beta}{\lambda}a'$. Even if the region encompassed by turbulent perturbations is measured in km, this fraction comprises only a few percent. Therefore dissipative energy in the column may be neglected. This is not a result which could be expected beforehand, since according to some data [12], atmospheric perturbations in a tornado reach the speed of sound.

Introducing Eq. (12) into Eq. (10), we define the Hankel transform for the desired temperature differential $\vartheta_0 = b\Theta s^{-1}J_1(rs)$. Substituting this function in Eq. (10), we obtain integral representations for the flow characteristics related to thermal convection. Finally (without consideration of dissipation)

$$\zeta = \zeta^{(1,1)} + \zeta^{(2)}, \quad \zeta^{(1,1)} = \frac{b\Theta}{2R}\int_0^\infty \frac{1}{s} J_1(bs) J_0(rs)\, ds; \quad (17)$$

$$p - R\zeta = p^{(1,1)} - R\zeta^{(1,1)} + p^2 - R\zeta^2, p^{(2)} - R\zeta^{(2)} = -\frac{\pi a^5}{288}\frac{2\left(a+|z|\right)^2 - r^2}{2\left[\left(a+|z|\right)^2+r^2\right]^{5/2}}, \quad (18)$$

$$p^{(1,1)} - R\zeta^{(1,1)} = -\frac{1}{2}b\Theta|z|\int_0^\infty e^{-s|z|}J_0(rs)J_1(bs)\,ds - \frac{b\Theta}{2}$$

(it is the difference $p - R\zeta$, and not the function p in accordance with Eq. (3) which characterizes the pressure in the medium);

$$w = w^{(1,1)} + w^{(2)}, \quad w^{(2)} = \frac{\pi a^5|z|}{1152}\frac{2\left(a+|z|\right)^2 - r^2}{\left[\left(a+|z|\right)^2+r^2\right]^{5/2}}, \quad (19)$$

$$w^{(1,1)} = -\frac{1}{8}b\Theta z\int_0^\infty \frac{1+s|z|}{s}e^{-s|z|}J_1(bs)J_0(rs)\,ds;$$

$$u = u^{(1,1)} + u^{(2)}, \quad u^{(2)} = \frac{\pi r a^5\left[r^2+\left(a+|z|\right)^2 - 3|z|\left(a+|z|\right)\right]}{1152\left[r^2+\left(a+|z|\right)^2\right]^{5/2}}, \quad (20)$$

$$u^{(1,1)} = -\frac{1}{4}b\Theta\int_0^\infty\left[\frac{|z|}{2s}\left(1+s|z|\right) - \frac{1}{s^2}\right]e^{-s|z|}J_1(rs)J_1(bs)\,ds;$$

$$T = b\Theta \int_0^\infty e^{-s|z|} J_1(bs)\, J_0(rs)\, ds. \tag{21}$$

The integral of Eq. (17) may be expressed in terms of different hypergeometric functions depending on whether r or b is the larger. Namely,

$$\zeta^{(1,1)} = \frac{b\Theta}{4r} \begin{cases} F\left(\frac{1}{2}, -\frac{1}{2}; 1; \frac{r^2}{b^2}\right), & r < b, \\ \frac{2}{\pi}, & r = b, \\ \frac{b}{2r} F\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{b^2}{r^2}\right), & r > b. \end{cases} \tag{22}$$

From this it follows that

$$\zeta^{(1,1)}(0) = \frac{b\Theta}{2R}, \quad \zeta^{(1,1)} = \frac{b^2\Theta}{4rR}\left[1 + O\left(\frac{b^2}{r^2}\right)\right], \qquad r \to \infty. \tag{23}$$

We will construct an asymptotic formula for the integral of Eq. (20) as b → ∞ by integrating the expression

$$\int_0^\infty e^{-\alpha s} J_1(bs)\, J_1(rs)\, ds = \frac{1}{\pi\sqrt{br}} Q_{1/2}\left(\frac{\alpha^2 + r^2 + b^2}{2rb}\right) = \frac{br}{2(\alpha^2 + r^2 + b^2)^{3/2}}$$

two times with respect to α from α to ∞. As a result we obtain

$$u^{(1,1)} = \frac{b^2 r\Theta}{16} \frac{(2r^2 + 2b^2 + z^2)\sqrt{r^2 + b^2 + z^2} - |z|(r^2 + b^2 + 2z^2)}{(r^2 + b^2 + z^2)^{3/2}\left(\sqrt{r^2 + b^2 + z^2} + |z|\right)}. \tag{24}$$

The remaining integrals in Eqs. (18), (19), (21) can be expanded in series of hypergeometric functions. The expressions obtained are no more obvious than the original ones, and will not be presented here.

We will note the major results and some consequences.

1. If there act upon the surface of the water or the surface of the density discontinuity in the atmosphere in the absence of temperature differentials stationary turbulent tangent stresses differing from zero in a region of radius a', where they are constant in value and equal to Ω, the dimensional characteristics of the flows produced can be obtained after transformation to the variables of Eq. (3) in Eqs. (13), (17)-(20), and (24) for Θ = 0:

$$v_r^{(2)} = xU \frac{x^2 + (y+1)^2 - 3y(y+1)}{[x^2 + (y+1)^2]^{5/2}}, \quad v_\theta^{(2)} = \frac{xV}{(1 + x^2 + y^2)^{3/2}}, \tag{25}$$

$$v_z^{(2)} = yU \frac{2(1+y)^2 - x^2}{[x^2 + (y+1)^2]^{5/2}}, \quad p'^{(2)} = p_* + \rho g a' y - \frac{2(y+1)^2 - x^2}{2[x^2 + (y+1)^2]^{5/2}} P,$$

$$\zeta'^{(2)} = \frac{1 - x^2}{(1 + x^2)^3}\zeta_*, \quad \zeta_{\min} = -\frac{\zeta_*}{27}, \quad x = \frac{r'}{a'}, \quad y = \frac{|z|}{a}, \quad 0 \leqslant x, y \leqslant \infty,$$

$$U = \frac{\pi\rho a'^3\Omega^2}{1152\mu^3} = \frac{\rho T^2}{1152\pi\mu^3 a'} = \frac{\rho M^2}{1152\mu^3 a' S}, \quad V = \frac{\Omega a'}{6\mu} = \frac{T}{6\pi\mu a'} = \frac{M}{6\mu S},$$

$$P = \frac{\pi\rho a'^2\Omega^2}{288\mu^2} = \frac{\rho T^2}{288\mu^2 S} = \frac{\pi\rho M^2}{288\mu^2 S^2}, \quad \zeta_* = \frac{\Omega^2 a'^2}{72 g\mu^2} = \frac{T^2}{72 g\pi\mu^2 S} = \frac{M^2}{72 g\mu^2 S^2},$$

where $S = \pi a'^2$ is the area over which the tangent stresses Ω are distributed, T = ΩS are the net transverse tangent stresses on the surface; $M = Ta'$ is the moment of the turbulent tangent stresses.

It follows from Eq. (25) that the vertical velocity in the vicinity of the axis is directed upward, opposite the force of gravity. In air columns without a temperature differential only ascending flows are possible in this region. The radial velocity in the column is equal to zero on the surface of a two-cavity hyperboloid of revolution $x^2 + (y+1)^2 = 3y(y+1)$. Within the lower cavity of this hyperboloid (the upper cavity does not intersect

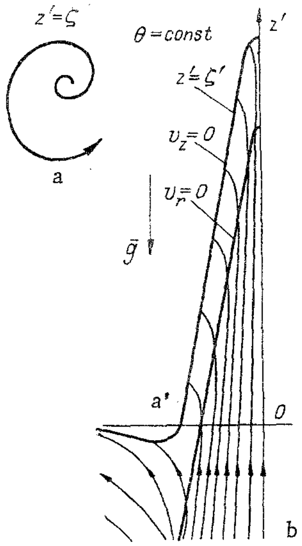

Fig. 1

the flow region) the radial velocity is directed toward the axis, while outside the cavity it is directed away from the axis of the turbulence. The flow lines on the fluid surface $z' = \zeta'$ are in the form of spirals $r' = C\exp\left(\frac{U}{V}\theta\right)$ (Fig. 1, above). Figure 1b shows an example of the flow line pattern and form of the surface $z' = \zeta'$ in the plane θ = const in the vicinity of the flow axis. The spatial configuration of the flow lines is close to that of a screw thread, along which flow proceeds toward the peak of the column. This is possible only if the height of the column increases constantly. The column is thus a mechanism for increase in its own height. Infinite increase in column height does not occur for the following reason. The hydrodynamic pressure on the surface $z' = \zeta'$ has a min $p' = p_* - P$ at $r' = 0$. In order that atomization of the liquid into the column not occur, it is necessary that $p_* \geqslant P$, since the liquid will not withstand negative pressure. From this and Eq. (25) it follows that motion in the column without liquid atomization is possible only under a load with a moment $M \leqslant 12\mu S\sqrt{2p_*/(\pi\rho)} = M_*$. For further increase in moment the oceanic waterspout begins to operate as a pump, pumping the liquid it atomizes into the atmosphere from its top [12]. A case is known where a waterspout carried off an entire lake into the atmosphere under such conditions [12]. Under dry land conditions with normal atmospheric pressure $M_*=0.09575$ SN·m (where S is in m²). The corresponding column height ζ_{**} and maximum transverse velocity $v_{\theta *}$ up to which fluid atomization in the column does not occur have the form $\zeta_{**} = \frac{4p_*}{\pi\rho g}$, $v_{0*} = 2\sqrt{\frac{2p_*}{\pi\rho}}$. For a column formed above water these values equal 13 m and 16 m/sec. In an air column, where the density decreases by almost 1000 times, the height limit ζ_{**} of the tornado is such that one cannot speak of any marked rarefaction of the air due to turbulent effects. The tornado lifts up the dense air of the lower atmospheric layers to a large height, and against the background of the rarefied air at this altitude the denser column is easily detected by radar equipment [13]. The dimensions of water surface perturbations and the velocity field in an oceanic waterspout can be judged from the following example. If at the water surface on an area S = 78.5 m² (a circle 10 m in diameter) there act turbulent tangent stresses with a moment M = 25 N·m, a column of atomized liquid with height $\zeta_* = 143.5$ m will be lifted from the surface. Ahead of this column there is formed a circular funnel 5.3 m in depth. The maximum transverse water velocity in such a spout is equal to 53 m/sec.

It follows from Eqs. (25) and (3) that the vertical velocity on the vortex axis $v(0,z') = yU(1 + y)^{-3}$ reaches a maximum $W = \pi a' g\zeta_*/(108\nu)$ at $z' = \zeta' - a'/2$. Even for perturbations ζ_* on the water surface of the order of several m the value of W exceeds the speed of sound by a factor of several times. This would be unreal under dry land conditions. Therefore in a more accurate approach it is necessary to construct a model of the column which considers compressibility of the fluid. It is interesting that a column without temperature differential practically neutralizes the effect of gravitation upon the liquid which it attracts. The pressure and velocity in the column are independent of the acceleration of gravity g.

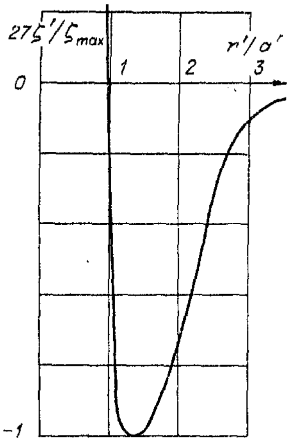

Fig. 2

2. In the approximation used here ($R \ll 1$) the arbitrary axisymmetric temperature perturbations of the density discontinuity surface do not distort the flow transverse velocity (Eq. (9)) over the entire region occupied by the fluid. This is related primarily to the fact that the Archimedean buoyant force produces no projection on the transverse axis.

3. If the temperature of the fluid density discontinuity surface is given by Eqs. (3), (12), we obtain the dimensional form of this surface after transition to the variables of Eq. (3) in Eqs. (16), (17), (22), and (23). The surface perturbation on the vortex axis

$$\zeta'(0) = \zeta_* + \frac{1}{2} b'\beta\Theta'. \tag{26}$$

If the ocean surface temperature is lower than that of the deeper water layers ($\Theta' < 0$) and if in Eq. (26) the second term is significantly greater than the first (which is possible only when the turbulent load at the surface is practically absent), then on this surface there is formed a hollow cavity with the profile of Eq. (22). The cooled, untwisted liquid surface is pressed into the ocean. Usually, even at low turbulent stress on the density discontinuity surface, the first term dominates in Eq. (26). The scale of surface deformation $z' = \zeta'$ in an air tornado due to heat exchange is equal to $|b'\Theta'\beta/2| \sim b'|\Theta'|/546$ ($\beta \sim 1/273\,°K^{-1}$), and for $b' \sim 10$ km, $|\Theta'| \sim 10\,°K$, is measured in tens or hundreds of m. The scale ζ_* of the deformation of this surface due to turbulent loading reaches tens of km (for air $\mu \sim 2 \cdot 10^{-5}$ kg/m·sec). Therefore the configuration of atmospheric perturbations in a tornado is characterized mainly by the turbulent term, Eq. (25), and appears as a tall narrow column (core) several tens of km high, surrounded in its lower portion by a narrow funnel 27 times shorter than the column height. (The funnel is narrow because the turbulent surface perturbations of Eq. (25) decrease with the fourth power of distance from the axis). This funnel itself has a depth of several hundred m. It can be observed visually and produces severe destruction at ground level [12]. Aside from visual observations, such a description of a tornado (ζ' in accordance with Eq. (25)) is supported by radar observations of such vortices [12, 13], which are noted as turbulent columns extending beyond the limits of the troposphere with a funnel at the base, many times shorter than the column height. Figure 2 shows a section of the funnel ahead of a column with density θ = const in accordance with Eq. (25). The funnel height is used as the vertical scale, while the horizontal scale is the radius of the zone of influence of turbulent perturbations, with the assumption that the first scale significantly exceeds the second.

4. The hydrodynamic pressure in the column with the temperature differential of Eq. (12) is described by Eqs. (3), (18). On the vortex axis and at its peak

$$p'(0, z') = p_* - \frac{\rho g^2 \mu^2}{\Omega^2}\left[\frac{\Theta - 2}{2}|z| + \frac{\Theta}{2}\left(b - \frac{z^2}{\sqrt{b^2 + z^2}}\right) + \frac{\pi a^5}{288\,(a + |z|)^3}\right],$$
$$p'(0, \zeta') = p_* - P - \frac{1}{2}\rho g b'\beta\Theta'. \tag{27}$$

From this it follows that in counterbalance to the general increase in vortex height a positive differential Θ' decreases, and a negative differential increases the pressure at the summit of the vortex. For example, heated ocean surfaces or cooled land surface decreases theheight of a waterspout or tornado up to the point of liquid atomization, i.e., the appearance of negative pressure in the column. On the other hand, a cooled body of water or heated land encourages increasein column height.

Equation (27) indicates that in the zone of tornado passage, especially if the earth is cooler than the cloud ($\Theta' > 0$), a region of reduced pressure must exist. This has been confirmed by observations (for example, storage containers explode, window glass is expelled outward, etc.). Without any detailed study of Eq. (27), we will indicate the possibility in principle of existence of positive roots of the equation $p'(0, |z|) = 0$. In this case the region of discontinuous pressure is located within the column. The presence of internal regions with discontinuous pressure implies the possibility of discontinuous tornadoes or waterspouts — isolated vortices separated by regions with a less dense medium. Such phenomena are sometimes observed in reality. If the equation $p'(0, |z|) = 0$ has no positive roots, then at a certain load on the density discontinuity surface (when Eq. (27) is negative) liquid atomization will occur at the top of the column. In this and other cases the column should be distinguishable from the planetary atomsphere when observed from space. If, as in the widely accepted hypothesis, the large red spot of the planet Jupiter is in fact a gigantic atmosphere vortex, (e.g., one generated by and breaking away from an equatorial jet), then the indicated effect may serve as an explanation of why this vortex, like the other smaller ones observed in the polar and other regions of Jupiter and Saturn [14] are detectable both by photography from space vehicles andvisual telescope observations.

5. The axial velocity in the column, defined by Eqs. (11), (12) on the density discontinuity surface, is given by Eqs. (3), (19), which on the flow axis produce

$$v_z(0, z') = \frac{yU}{(1+y)^3} + \frac{y_1 W_1}{\sqrt{1+y_1^2}}, \quad y_1 = \frac{|z|}{b}, \quad W_1 = \frac{\rho g \beta \Theta' b'^2}{8\mu}. \tag{28}$$

From this it follows that in the case of positive temperature differential the axial velocities produced by turbulent atmospheric perturbations and thermal convection add together. Thus, e.g., if the earth is colder than the cloud cover, then within the tornado (i.e., in the region $z' < \zeta'$ below the density discontinuity surface) only ascending flows will be found.

For a negative temperature differential at small $|z|$ the velocity in Eq. (28) is positive (not only because in practice $U > |W_1|$, but also because usually $b' \gg a'$) and therefore directed toward the top of the vortex. As $z' \to -\infty$ the first term in Eq. (28) tends to zero, and the second to $W_1 < 0$. Therefore, far from the top of the tornado the axial velocity is oriented downward into the fluid. The change in direction of the velocity of Eq. (28) with increase in $|z|$ occurs sooner, the more accurately the inequality $b' \gg a'$ is satisfied, i.e., when the temperature differential region is significantly wider than the region of action of turbulent perturbations. This coincides with the conditions for tornado formation: The air column develops when a large cloud passes over the land and turbulence breaks off from the cloud edges in the form of a closed cloud, which serves as the seed for the tornado [6, 13]. The large surface of the cloud ensures the existence of a large region of temperature difference between cloud and earth, while the relative smallness of the turbulence breaking off from the cloud edges ensures a high narrow column and deep narrow tornado funnel. Then at the base of the vortex axial velocities which are related to thermal convection predominate over those produced by the vortex itself. If the earth is warmer than the surrounding air, then intense descending flows will be observed along the tornado axis, produced by the high pressure on the earth. This is connected with the pressing into the earth of various objects which sometimes occurs upon passage of a tornado. Until the present time, this property of an atmospheric vortex could only be explained by recourse to a number of electromagnetic hypotheses [2].

6. The radial velocity in the column is expressed by Eqs. (3), (20). On the density discontinuity surface

$$v_r(r', \zeta') = \frac{xU}{(1+x^2)^{3/2}} + \frac{x_1 W_1}{\sqrt{1+x_1^2}}, \quad x_1 = \frac{r'}{b'}.$$

With a positive temperature differential the flow pattern in the plane θ = const is analogous to that shown in Fig. 1b. In the case of a negative differential, far from the vortex axis convection flows predominate, directed toward that axis. These flows are divided by a surface $v_z(r', z') = 0$ into two regions, in one of which the flow encourages increase in column height, while in the other a descending flow is created.

7. The temperature differential in the flow zone is defined by Eqs. (3), (21). On the vortex axis and at $r' \to \infty$

$$T'(0, z') = \frac{\Theta'}{1 + y_1^2 + y_1 \sqrt{1 + y_1^2}}, \quad T'(r', z') = \frac{b'^2 |z'| \Theta'}{2(r'^2 + z'^2 + b'^2)^{3/2}}, \qquad r' \to \infty. \tag{29}$$

8. We will present some considerations which permit extension of the results of Eq. (25), which characterizes flow of $a = a'\Omega^2/(g\mu^2)$, to the region of high values for this parameter. Equation (25) was obtained from Eq. (13) obtained in turn from the integral

$$v = \frac{1}{\pi \sqrt{r}} \int_0^a \sqrt{x}\, Q_{1/2}\left(\frac{x^2 + r^2 + z^2}{2rx}\right) dx$$

as $a \to 0$. As $a \to \infty$ this integral diverges [15]. For a qualitative estimate of its growth as $a \to \infty$ the Legendre function may be replaced by its asymptote for large values of the argument, since its dependence on this argument as $x \to 0$ and $x \to \infty$ is described by one and the same formula. This gives

$$v \sim \frac{r}{2} \int_0^a \frac{x^2 dx}{(x^2 + r^2 + z^2)^{3/2}} \sim \frac{r}{2} \ln \frac{2a}{\sqrt{r^2 + z^2}}, \qquad a \to \infty.$$

Consequently, at large a the transverse velocity in the column grows logarithmically with a. Equation (13) used above is finite as $a \to \infty$. Therefore, by using Eq. (25) at large values of a, we can only reduce the absolute values of the extremal characteristics of the flow under consideration.

We note that in constructing the tornado model possible turbulence of the flow under consideration was ignored. There exists a numerical calculation of a tornado [16] with and without consideration of turbulence, which found no significant differences between the laminar and turbulent cases. It is possible that these vortices possess a unique mechanism for suppression of turbulence, which justifies the laminar model used here.

9. The results obtained may be extended to the case of motion of the medium under the action of turbulent and temperature perturbations moving over the density discontinuity surface with a specified constant velocity c'. Let the values ω', ϑ' depend solely on the distance r' from the center of the perturbations. Then, writing the nonstationary system of Navier–Stokes and thermal balance equations in the variables $r' = \sqrt{(x' - c't')^2 + y'^2}$, $\tilde{x} = x' - c't'$ z' and using the fact that the boundary conditions are dependent only on r', we will seek a solution of the system which is independent of $\tilde{x}$. This solution can be obtained by the previously used replacement $r' = \sqrt{(x' - c't')^2 + y'^2}$ and contains information on the thermo- and hydrodynamic wake of the column in the atmosphere and ocean. Without any detailed analysis of this information, we will note that no limitations were placed above on the radius b' of the zone of action of temperature perturabations abovethe density discontinuity surface. Therefore, at sufficiently large b' Eq. (29) may be used for analysis of the temperature wake of a hurricane above a water surface (being limited to the simplest assumption that the hurricane, the axis of which translates in one direction at constant velocity c', createsin a fixed circle of radius b' with center on this axis a constant temperature differential Θ' between the water surface and the deep layers of the hurricane). From Eq. (29) we write the equations of the isotherms $T' = \text{const}$ as $r' \to \infty$ in the form

$$\sin^2 \varphi = \frac{(\rho^2 + 1)^3}{\rho^2} \frac{4T'^2 (b'^2 + \tilde{x}^2)^2}{b'^4 \Theta'^2}, \quad \begin{aligned} y' &= \rho \sqrt{b'^2 + \tilde{x}^2} \cos \varphi, \\ z' &= \rho \sqrt{b'^2 + \tilde{x}^2} \sin \varphi. \end{aligned} \tag{30}$$

The condition for existence of isotherms is the inequality $\sin^2\varphi \leqslant 1$, which in view of Eq. (30) reduces to study of the case where the third-order polynomial in $\rho^2 + 1$ has two roots not less than unity. This places a limitation on the coefficients of the polynomial, from which it follows that

$$|\tilde{x}| = |x' - c't'| \leqslant b' \sqrt{\frac{1}{3\sqrt{3}}\left|\frac{\Theta'}{T'}\right| - 1}. \tag{31}$$

From this it follows that at large values of time t' the isotherms are significant only for $3\sqrt{3}|T'| \leqslant |\Theta'|$. The relaxation time t'_* of the hurricane wake to temperature T' in the section x' = 0 can be obtained if in Eq. (31) we take the equality sign: $t'_* = \frac{b'}{c'}\sqrt{\frac{1}{3\sqrt{3}}\left|\frac{\Theta'}{T'}\right| - 1}$. For $t' = t'_*$ the isotherms of Eq. (30) degenerate into a point, disappearing for $t' > t'_*$. Thus, the relaxation time of a hurricane or tornado thermal wake is directly proportional to the radius of the zone of temperature perturbations of the density discontinuity surface, and inversely proportional to the translation velocity of these perturbations. If the initial temperature differential Θ' = 10°C, the radius of the temperature perturbation zone b' = 2000 km (neutral cyclone stage of [12]), and the translational velocity of the cyclone c' = 1 m/sec, then the relaxation time of the hurricane wake to a temperature T' = 1°C t'_* = 22 days. This result indicates very slow thermal relaxation of a hurricane wake. It agrees with latter data from the 27th voyage of the observation vessel Akademik Kurchatov participating in the joint Soviet—American program Polymode [17]. During this voyage the thermal wake of a relative weak hurricane in the Sargasso Sea was charted one day and 20 days after its formation. Study of the resulting isotherms indicated that in 20 days the relaxation of the thermal wake to a temperature of 1°C had practically not begun. The isotherms of Eq. (30) in the plane yOz appear as ovals with an axis of symmetry y = 0. Analysis of experimental isotherms of a hurricane thermal wake [17] shows that in the process of deformation there is a tendency for degeneration of the isotherms into ovals.

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CUMULATION EFFECT IN DYNAMIC PRESSING OF POWDERED MATERIALS

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One of the promising methods of pressing powdered materials is the passage of a high-density electrical current through the powder [1, 2]. This method produces high-density materials with required characteristics. The density of the pressed materials is controlled by choice of pressing parameters: the mechanical loading applied, and the amplitude and duration of the current pulses. It has been established experimentally that certain parameter values exist, at which the pressing process becomes unstable — "spattering" of material from the press form occurs [1]. The present study will consider the possible cause of such "spattering" and define the range of parameter values within which this phenomenon occurs. The behavior of powdered material subjected to compression by applied pressure can be described with the aid of the "hollow sphere" model [3]. At values of the deformation rate tensor components in the range 10^3–10^5 sec^{-1} the rheological behavior of the powder material corresponds quite well to that of a viscoplastic material with hardening [3, 4]. In this case the equation describing the change in porosity of the pressed material $\alpha = v/v_m$, where v is the specific volume of the powder and v_m is the specific volume of the bulk material forming the powder ($\alpha > 1$), has the form [3]

$$-\frac{1}{3}(\alpha_0-1)^{-2/3}\frac{d}{d\alpha}\left\{\frac{\dot{\alpha}^2}{2}\left[(\alpha-1)^{-1/3}-\alpha^{-1/3}\right]\right\}=1-\frac{2}{3}\beta\times$$
$$\times\left\{\ln\frac{\alpha}{\alpha-1}+3m\int\limits_1^{\left(\frac{\alpha}{\alpha-1}\right)^{1/3}}\left[\frac{2}{3}\ln\left(1+\frac{\alpha_0-\alpha}{(\alpha-1)x^3}\right)\right]^n\frac{dx}{x}\right\}+\frac{4}{3\,\mathrm{Re}_0}\frac{\dot{\alpha}}{\alpha(\alpha-1)}, \quad (1)$$

where $\mathrm{Re}_0 = (a_0/\nu)\sqrt{p/\rho}$; $\beta = Y_0/p$; $\tau = a_0\sqrt{\rho/p}$; a_0 is the characteristic size of the pores, ν is viscosity, ρ is the density of the powder material, p is the external pressing pressure, α_0 is the initial porosity value. The dot indicates differentiation with respect to dimensionless time t/τ. The material hardening law is chosen in the form [3]

$$Y = Y_0\left(1+m\left(\bar{\varepsilon}^p\right)^n\right),$$

where Y_0 is the initial yield point; m, n are hardening parameters; $\bar{\varepsilon}^p$ is the accumulated plastic deformation.

Depending on the value of the parameters Re_0 and β, Eq. (1) produces two qualitatively different types of solution $\alpha(t)$: the first consists of solutions defining a finite porosity value for pressing with $\alpha > 1$, while the second produces a final porosity of $\alpha = 1$ (poreless material) with $\dot{\alpha} \neq 0$ (at the moment $\alpha = 1$). The solutions of the second type can be analyzed conveniently by commencing from the corresponding equation for change in internal radius of a "hollow sphere", $a(t)$, which can be obtained from Eq. (1), considering that

$$\xi = \frac{a(t)}{a_0} = \left(\frac{\alpha(t)-1}{\alpha_0-1}\right)^{1/3}, \quad u = \frac{da}{dt}\sqrt{\frac{\rho}{p}},$$

where u is the dimensionless rate of motion of the internal radius of the "hollow sphere" and ξ is the dimensionless internal radius of the same "hollow sphere."

With such notation Eq. (1) takes on the form

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